Friedman Cosmological Models with

both Radiation & Matter

In a recent paper, an analytic solution to Einstein's field equations was given.[1] This solution represents a homogeneous & istropic universe containing both radiation and dust; the space-like surface (on which the matter is comoving) is flat. However, using arguements based on Mach's principle, Wheeler[2] argues that space is closed. This implies for a Friedman universe that the curvature is positive. On the other hand, using statistical arguements and observational data, H-Y Chiu[3] argues that the universe can be described by an open Friedman model with negative curvature. This disagreement cannot be resolved at this time since the energy density of the universe is not known to sufficient accuracy. In view of this, it seems reasonable to consider all three Friedman cosmological models.

A universe which is spatially homogeneous & isotropic but not necessarily flat can be described by the Robertson-Walker metric[4]:

$$ds^2=-dt^2+a^2(t)\left[1+(k/4)u^2\right]^2\left[du^2+u^2(d_\theta{}^2+\sin^2{}_\theta d\phi^2)\right] \qquad (1)$$

where $k=1,0,-1$ corresponds respectively to positive, zero, negative curvature of the spacelike surface $t=$ constant. We also assume that the radiation & dust expand independently & adiabatically and that $(8\pi/3)\ \rho_m = K_m\ a^{-3}$ for the dust and $\rho_R = 3p_R = 3K_r a^{-4}/8\pi$ for the radiation where K_m and K_r are constants. This choice satisfies the conservation law

$$T^{\mu\nu}{}_{;\nu} = 0 = (\rho a^3)^{\bullet} + p\,(a^3)^{\bullet}$$

where the dot denotes differentiation with respect to time, ρ is the energy density, and p is the pressure. The only remaining equation to be solved is

$$\dot{a}^2 + k = K_m\ a^{-1} + K_R\ a^{-2}.$$

If this equation & the conservation law are satisfied, the other Einstein equation is satisfied automatically.

The three solutions are:

$$t - t_o = (K_R + K_m a - a^2)^{\frac{1}{2}} + (K_m/2)\ \sin^{-1}\left[\ (K_m - 2a)\ \ (K^2{}_m + 4K_r)^{\frac{1}{2}}\right]$$

for $k=1$ positive curvature,

$$t - t_o = 2\,(K_m a - 2K_R)\ \ (K_r + K_m a)^{\frac{1}{2}}/3K_m{}^2$$

for $k=0$ zero curvature,

$$t - t_o = (K_R + K_m a + a^2)^{\frac{1}{2}} - (K_m/2)\ \ln\left[(K_R + K_m a + a^2)^{\frac{1}{2}} + a + (K_m/2)\right]$$

for $k=-1$ negative curvature.

For completeness we gave all three solutions. The requirement that $a=0$ at $t=0$ fixes the integration constant t_o.

The hubble constant $H=\dot{a}/a$ and the deceleration parameter $q=-\ddot{a}/aH^2$ are related to the total energy density and pressure via

$$3qH^2 = 4\pi(\rho+3p)$$

Using the total energy density and the energy density of the radiation, one can (in principle) find the curvature constant k via

$$ka^{-2} = (8\pi/3)\,\dot{\rho}-H^2.$$

However, because of the uncertainty in the density, there is no general agreement on the value of k[1,2,3]. The situation is complicated still further since the mass density necessary for the binding of our cluster of galaxies is much larger than that which is observed[5].

Jeffery M. Cohen

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Footnotes

1. K. Jacobs, Nature 215, 1156 (1967)

2. J.A. Wheeler, Gravitation & Relativity (Edited by Chiu & Hoffmann) Benjamin (New York) 1964

3. H.Y. Chiu, Ann Phys. 43, 1 (1967)

4. Robertson, Ap. J. 82, 284 (1935)

   Walker, Proc. Lon. Math. Soc., Ser.2, 42, 90(1937)

5. For a discussion of cosmological solutions and astronomical observations see e.g. article by O. Heckmann and E. Schucking in Gravitation (Edited by L. Witten) Wiley, New York (1963) and the references cited there.